MATH 1207, Honors Math A Notes

Contents

1 Suprema

Definition. A subset $S \subseteq \mathbb{R}$ is bounded above if for each $y \in \mathbb{R}$ such that for all $x \in S, x \leq y$.

Definition. An upper bound y for $S \subseteq \mathbb{R}$ is a least upper bound or supremum if y is an upper bound for S and if z is an upper bound for S, then $y \leq z$.

We notate $\sup(S)$ as the supremum of S.

Axiom. (Completeness) Every nonempty bounded above set of real numbers has a supremum.

Note that this is false for $\mathbb{Q}$. You can prove directly for $\sqrt{2}$ for a hard exercise.

Remark. $\mathbb{R}$ is unique up to isomorphism.

Prop. *Suprema are unique; if y, y' are sumprema, then we have that $y = y'$.*

Proof. This is proved that if y is a supremum, and y' is another upper bound, then $y \leq y'$. Similarly, we have that $y' \leq y$. By trichotomy, it follows that they are equal. □

Definition. S is bounded below if $\exists y \mid \forall x \in S, y \leq x$. That y is a lower bound for S. Then y is the greatest lower bound, or the infimum, if y is a lower bound and if z is any lower bound, then $z \leq y$.

We notate $\inf(S)$ as the infimum of S.

Prop. *If $S \subseteq R, S \neq \emptyset$ is bounded below, then $\exists! \inf(S)$.*

Proof. Let $-S = \{-x \mid x \in S\}$. Then $-S$ is nonempty and bounded above. Then $\sup(-S)$ exists. Then existence and uniqueness follows from the claim that $\sup(-S) = \inf(S)$. □

Theorem. *(Approximation) Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. $\forall \epsilon > 0, \exists x \in S \mid \sup(S) - \epsilon < x$.*

Proof. We proceed via contradiction. Suppose that $\exists \epsilon \mid \forall x \sup S - \epsilon \geq x$. Then $\sup S - \epsilon$ is an upper bound for S. By definition of sup, we have the statement $\sup \leq \sup(S) - \epsilon$ but as $\epsilon > 0$. $\Rightarrow\!\Leftarrow$ □

Theorem. *(Additivity of suprema) If we have $S, T \subseteq \mathbb{R}$ nonempty and bounded above, let $S + T := \{s + t \mid s \in S, t \in T\}$. Then $\sup(S + T)$ exists and equals $\sup(S) + \sup(T)$.*

Proof. Let $s = \sup(S), t = \sup(T)$. Then $\forall x \in S, x \leq s, \forall y \in T, y \leq t \implies \forall x \in S, \forall y \in T, x + y \leq s + t \implies s + t$ is an upper bound for $S + T$.

Suppose that $s + t$ is not the least upper bound. Then $\exists \delta > 0 \mid s + t - \delta$ is an upper bound for $S + T$. Let $\epsilon = \frac{\delta}{2}$. Then by the theorem regarding approximation, we have that $\exists x \in S \mid s - \epsilon < x, \exists y \in T | t - \epsilon < y$. Then $x + y \in S + T$. Further, $s + t\delta = s + t - 2\epsilon \leq x + y$. $\Rightarrow\!\Leftarrow$ □

Prop. *Suppose we have nonempty $S, T \subseteq \mathbb{R}$ such that $\forall x \in S, \forall y \in T, x \leq y$. Then $\sup(S), \inf(T)$ exist, and $\sup(S) \leq \inf(T)$*

Proof. Any $x \in S$ is a lower bound for $T \implies \exists \inf(T)$. Similarly, any $y \in T$ is an upper bound for $S \implies \exists \sup(S)$.

Suppose that $\sup(S) > \inf(T)$. Then let $\delta = \sup(S) - \inf(T)$. Let $\epsilon = \frac{\delta}{2}$.

Then by approximation, we have that $\exists x \in S \mid \sup(S) - \epsilon < x$. Similarly, we have that $\exists y \in T \mid \inf(T) + \epsilon > y$.

This yields that $y < \inf(T) + \epsilon = \sup(S) - \epsilon < x$. $\Rightarrow\!\Leftarrow$ □

Theorem. *$\mathbb{N}$ has no upper bound.*

Proof. Suppose that $\mathbb{N}$ does in fact have an upper bound. Let this bound be Ψ. The approximation property with $\epsilon = \frac{1}{2}$ implies that we can find a $n \in \mathbb{N}$ such that $\Psi - \frac{1}{2} < n$. However, $n + 1 > \Psi \in \mathbb{N}$. $\Rightarrow\!\Leftarrow$ □

Definition. For a function $|\cdot| : \mathbb{R} \to \mathbb{R}$,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Note that this gives us a natural way to define distance, removing the casework needed when we need to refer to such concepts. An example is the following:

Theorem. *(Triangle Inequality)* $|x + y| \leq |x| + |y|$.

Proof. Equality holds if $x, y = 0$, $x, y > 0$, $x, y < 0$.

If $x < 0, y > 0$, then $|x| + |y| = x - y > -x - y, x + y$.

If $x > 0, y < 0$, then it follows as the last step is symmetric. □

Corollary. $|x - z| \leq |x - y| + |y - z|$.

(Mentorship program for math, bit.ly/AWMMentorship2019)

Prop. *(Archimedian property) Let $x > 0$ and $y \in \mathbb{R}$. Then $\exists n \in \mathbb{N}$ such that $nx > y$.*

Proof. This follows from the fact that $\mathbb{N}$ has no upper bound, which means that $\exists n \in \mathbb{N}$ larger than $\frac{y}{x} \iff nx > y$. $\square$

Corollary. *If $a, x, y \in \mathbb{R}$ with $a \leq x \leq a + \frac{y}{n} \forall n \in \mathbb{N}$, then $a = x$.*

Proof. Assume otherwise, so $a < x \iff x - a > 0$. The Archimedian principle states that $\exists n \in \mathbb{N} \mid n(x - a) > y$. However, we had previously that no such n exists. $\Rightarrow\Leftarrow$ $\square$

2 Finite and Infinite Sets

Notation: $[n] = \{m \in \mathbb{N} \mid 0 < m \leq n\}$

Theorem. *$\forall m, n \in \mathbb{N}$, the following are true:*

$$a)\ \exists f : [m] \to [n] \iff m \leq n,\ f \text{ injective}$$
$$b)\ \exists f : [m] \to [n] \iff m \geq n,\ f \text{ surjective}$$

Proof. (a, $\Longleftarrow$) define $f : [m] \to [n]$ such that $f : i \mapsto i$. f is clearly injective.

(a, $\Longrightarrow$) We proceed by inducting on m. The base case of $m = 0$ is true on inspection from the properties of the empty set and empty function.

Suppose that this holds for a given m and all n. Suppose $f : [m + 1] \to [n]$ is injective. $\forall i \in [m + 1], i \neq m + 1 \implies f(i) \neq f(m + 1)$.

Define $\bar{f} : [m] \to [n] \setminus \{f(m + 1)\}, \bar{f}(i) = f(i)$. This is still injective. We now define $h : [n] \setminus \{f(m + 1)\} \to [n - 1]$.

$$h(i) = \begin{cases} i & i < f(m+1) \\ i - 1 & i > f(m+1) \end{cases}$$

This is easily shown to be injective. Further, $h \circ \bar{f} : [m] \to [n - 1]$, which is injective as the composition of two injective functions. The inductive hypothesis yields that $m \leq n - 1 \implies m + 1 \leq n$.

(b) is very similar, so the proof is omitted. $\square$

Definition. A set S is finite if there exists a bijection $f : [n] \to S$ for some $n \in \mathbb{N}$.

Definition. If a set S is not finite, it is infinite.

Prop. *Given a finite set S, the $n \in \mathbb{N}$ as above is unique.*

Proof. Suppose that we have two bijections $f, g[n] \tilde{\to} S$. Then let h: $S \tilde{\to} [n]$ be the inverse of f. Then $h \circ g : [m] \tilde{\to} [n]$ is bijective.
The previous theorem applied to $h \circ g$ implies that $m \leq n, n \leq m \implies n = m$. □

We notate this n as $|S|$.

Example. For finite sets S, T, then $S \cup T, S \cap T, S \times T, S^T$ are finite.

Example. Any subset of S is finite, and if $S \subseteq T$ then $|S| \leq |T|$.

Example. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are infinite sets.

Definition. Suppose $f : S \to T$, $U \subseteq S$. Then the restriction of f to U, $f|_U : U \to T, f|_U : s \mapsto f(s)$.

Definition. The inclusion of $S \subseteq T$ is $\mathrm{id}_T|_S$.

Definition. For $a, b \in \mathbb{R}$, an interval is one of

$$[a, b] = \{a \leq x \leq b \mid x \in \mathbb{R}\}$$
$$(a, b) = \{a < x < b \mid x \in \mathbb{R}\}$$
$$[a, b) = \{a \leq x < b \mid x \in \mathbb{R}\}$$
$$(a, b] = \{a < x \leq b \mid x \in \mathbb{R}\}$$

We also allow a, b to be ∞ or $-\infty$ for open intervals with $-\infty < x < \infty, \forall x \in \mathbb{R}$.

Example. If $a \neq b$, these are infinite sets.

3 Real-Valued Functions

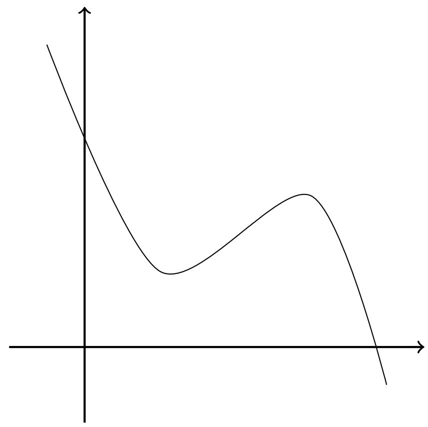
Definition. If $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$, then $f + g : [a, b] \to \mathbb{R}, fg : [a, b] \to \mathbb{R}$ are functions defined by $(f + g)(x) = f(x) + g(x), (fg)(x) = f(x)g(x)$.
All the above are real-valued functions.

Definition. Similarly, if $c \in \mathbb{R}$, $cf : [a, b] \to \mathbb{R}, (cf)(x) = cf(x)$.

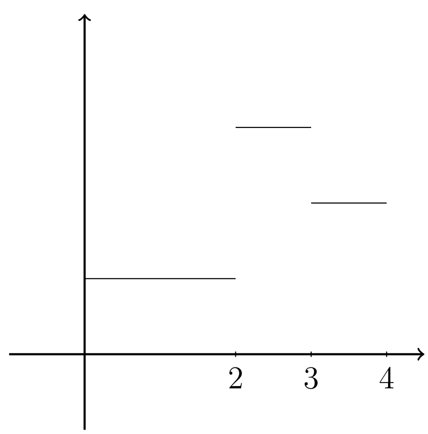
Definition. If $f_1, f_2, ..., f_n : [a, b] \to \mathbb{R}$, $c_1, c_2, ..., c_2 \in \mathbb{R}$ then the corresponding linear combination is

$$(\sum_{i=1}^{n} c_i f_i)(x) = \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i f_i$$

4 Step Functions



Definition. $f : [a, b] \to \mathbb{R}$ is a step function if there is a finite set of real numbers $S = \{x_0, ..., x_n\} \subset \mathbb{R}$ called a partition with $a = x_0 < x_1 < ... < x_n = b$, and $c_1, ..., c_n \in \mathbb{R}$ such that $\forall i \in [n], \forall x \in (x_{i-1}, x_i), f(x) = c_i$. This is equivalent to $f|_{(x_{i-1}, x_i)} = c_i$.



Prop. *If $f, g : [a, b] \to \mathbb{R}$ are step functions, so are $f + g, fg$.*

Proof. Let S be a partition for f, T a partition for g, then $S \cup T$ is a partition for $f + g, fg$. More specifically, let $S = \{x_0, ..., x_m\}, T = \{y_0, ..., y_n\}$. Further, $S \cup T = \{z_0, ..., z_p\}$ and is finite.

For any $z_k \in S \cup T$ with $k > 0$, let x_i be the greatest element of S such that $x_i < z_k$, y_j the greatest element of T such that $y_j < z_k$.

Then $z_{k-1} = \max(x_i, y_j), z_k = \min(x_{i+1}, y_{j+1})$. Hence $(z_{k-1}, z_k) \subset (x_i, x_{i+1}) \cap (y_i, y_{j+1})$.

We can then see that $fg, f + g$ are constant on these intervals. □

5 Integrals of Step Functions

We have previously defined step functions, and now we define their integrals:

Definition. Let f be a step function on $[a, b]$ with partition $\{x_0, x_1 ... x_n\}$ and such that $f|_{(x_{i-1}, x_i)}(x) = c_i$ with $c_i \in \mathbb{R}$. Then,

$$\int_a^b f = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$$

.

This can also be notated as $\int_a^b f(x)dx$.

Prop. *This is well-defined, not depending on partition.*

Proof. Suppose that we are given by two partitions P, Q.
First suppose that $P \subseteq Q$. Use that $c(x_{i+2} - x_i) = c(x_{i+2} - x_{i+1}) + c(x_{i+1} - x_i)$, and induct on the amount of points added, or $|Q - P|$.
In general, for any two partitions P, Q, we notice that P is contained in $P \cup Q$ and also Q is contained in $P \cup Q$. Applying the first case twice, we have that these are still equivalent. □

We have a few conventions here: $\int_a^a f := 0$. If $b < a$, $\int_a^b f = -\int_b^a f$.
There are a long list of properties that are satisfied by the integral of step functions.

Theorem. *(Properties of $\int$ for step functions). Let $f, g : [a, b] \to \mathbb{R}$ be step functions. Then we have the following:*

1. $\int_a^b (f + g) = \int_a^b f + \int_a^b g$
2. $\int_a^b cf = c \int_a^b f$
3. $\int_a^b \sum_{i=1}^n c_i f_i = \sum_{i=1}^n c_i \int_a^b f_i$
4. *If $f \leq g$, or $\forall x \in [a, b], f(x) \leq g(x)$, then $\int_a^b f \leq \int_a^b g$.*
5. $\int_a^c f = \int_a^b f + \int_b^c f$
6. *$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x - c)dx$, or congruently, we have that $g : [a + c, b + c] \to \mathbb{R}, g(x) = f(x - c)$, then the latter is $\int_{a+c}^{b+c} g$.*
7. *If $c \neq 0$, then $\int_{ca}^{cb} f(\frac{x}{c})dx = c \int_a^b f(x)dx$.*

Proof. (Additivity) If P, Q are partitions for f, g, then $P \cup Q$ is a partition for f and g. Say $f|_{(x_{i-1},x_i)}(x) = c_i, g|_{(x_{i-1},x_i)}(x) = d_i$.
Then $(f+g)|_{(x_{i-1},x_i)}(x) = c_i + d_i$, so

$$\begin{aligned}
\int_a^b f + g &= \sum_{i=1}^{n} (c_i + d_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (c_i(x_i - x_{i-1}) + d_i(x_i - x_{i-1})) \\
&= \sum_{i=1}^{n} c_i(x_i - x_{i-1}) + \sum_{i=1}^{n} d_i(x_i - x_{i-1})
\end{aligned}$$

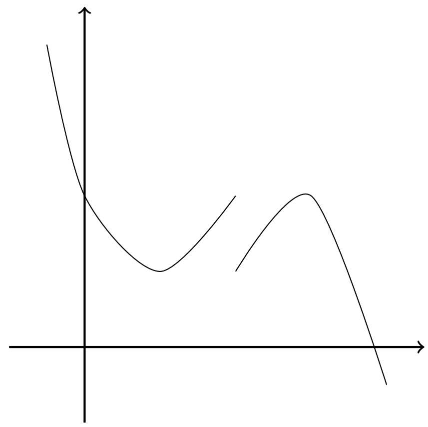
Justifying the last step, we can induct on n. The base case of $n = 0$ has everything empty, so the claim follows in that case.

The inductive step is as follows: assume that for n,

$$\begin{aligned}
\sum_{i=1}^{n+1} (a_i + b_i) &= \sum_{i=1}^{n} (a_i + b_i) + a_{n+1} + b_{n+1} \\
&= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i + a_{n+1} + b_{n+1} \\
&= \sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n+1} b_i
\end{aligned}$$

□

6 Integrals on More General Functions



Definition. Let $f : [a,b] \to \mathbb{R}$ be a function. Then f is bounded if $\exists C \in \mathbb{R}$ such that $|f(x)| \leq C \forall x \in [a,b]$.

Definition. For f bounded, the lower integral is $\underline{I}(f) = \sup \underline{S}(f)$, where $\underline{S}(f) = \{\int_a^b s \mid s \leq f\}$, with s a step function.

Definition. For f bounded, the lower integral is $\overline{I}(f) = \inf \overline{S}(f)$, where $\overline{S}(f) = \{\int_a^b s \mid s \geq f\}$, with s a step function.

Prop. *This is well defined.*

Proof. For $\underline{I}(f)$, we need to check that $\underline{S}(f)$ is not empty and bounded above.

This is nonempty, as the constant function $s(x) = -C$ is a step function that is constant and $s \leq f$.

This is bounded above, as we have an upper bound of $\int_a^b t$, where $t(x) = C$. This follows as $f \leq t$, we have that any step function s with $s \leq f$ satisfies $s \leq t$.

By comparison of step functions, $\int_a^b s \leq \int_a^b t$. Thus, the supremum exists. Similarly, the infimum also exists. □

Definition. If $f : [a,b] \to \mathbb{R}$ is a bounded function, we say that f is integrable if $\underline{I}(f) = \overline{I}(f)$, and the integral is $\int_a^b f = \underline{I}(f) = \overline{I}(f)$.

Prop. $\underline{I}(f) \leq \overline{I}(f)$.

Proof. $\forall s, t$ with $s \leq f \leq t$, we have that $\int_a^b s \leq \int_a^b t$ by comparison of step functions. So, $\forall x \in \underline{S}(f), y \in \overline{I}(f), x \leq y \implies \sup \underline{S}(f) \leq \inf \overline{S}(f)$. □

Remark. This is known as a Riemann integral. There are also different types of other integrals, such as Lebesgue integrals, but they will not be used in this class.

7 Nonintegrable Functions

Theorem. *Let $f : [0, 1] \to R$ be defined*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

f is not Riemann integrable. (This assumes the existence of irrationals, but this is shown in Apostol (such as $n-th$ roots of positive reals), or by results on cardinality.

Proof. A homework problem implies that $\forall a < b \in \mathbb{R}, \exists x \in \mathbb{Q} \mid a < x < b$, or that $\mathbb{Q}$ is dense in $\mathbb{R}$.

There is some irrational $x \in \mathbb{R}$. Without loss of generality, take $x > 0$ (otherwise, take $-x$). The archimedian property implies that $\exists n \in \mathbb{N}$ with $n > x$. Thus, $0 < x < n \implies 0 < \frac{x}{n} < 1 \implies 0 < (b-a)\frac{x}{n} < b - a \implies a < (b-a)\frac{x}{n} + a < b$.

Now, the above works for $a, b \in Q$, since $\mathbb{Q}$ is a field, we have that if $(b-a)\frac{x}{n} + a$ were rational then so would x, so $(b-a)\frac{x}{n}$ is irrational.

Now suppose $a < b$ are arbitrary reals. The density of $\mathbb{Q}$ implies $\exists c, f \in \mathbb{Q}, a < c < d < b$. Apply c, d to the previous step, and we have an irrational x with $c < x < d \implies a < x < b$. Thus, $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

To calculate the lower integral, let s be a step function with $s \leq f$. By definition, $\exists$ partition $P = \{x_0, ...x_n\}$ with $S|_{(x_{i-1}, x_i)}$ constant. However, $\exists y \in \mathbb{R} \setminus \mathbb{Q}$ with $x_{i-1} < y < x_i \implies f(y) = 0 \implies s(y) \leq 0 \implies S|_{(x_{i-1}, x_i)} \leq 0 \implies s \leq 0$ except at points of P. Hence, $\int_0^1 s \leq 0 \implies \underline{I}(f) \leq 0$. In fact, equality actually holds, but this is not needed for the proof.

Similarly, we have that if $t \geq f$, $t|_{(x_{i-1}, x_i}$ constant on subintervals of some partition, we have that $\exists x \in \mathbb{Q} \mid x_{I-1} < x < x_i \implies f(x) = 1 \implies t(x) \geq 1 \implies t|_{(x_{i-1}, x_i} \geq 1 \implies t \geq 1$, except at points of the partition. Hence, $\overline{I}(f) \geq 1$. In fact, equality actually holds, but this also is not needed for the proof.

Since $\underline{I}(f) \neq \overline{I}(f) \implies f$ is not integrable. □

8 Integrable Functions

Lemma. *If $S \leq T$, both bounded above, nonempty in $\mathbb{R}$, then* $\sup(S) \leq \sup(T)$.

Proof. $\sup(T)$ is an upper bound for S, so $\sup(S) \leq \sup(T)$. □

Theorem. *All previous properties of $\int_a^b$ for step functions also hold for integrable functions. Also, if the functions f, g, f_i are integrable, so are $f + g, fg$, etc.*

Proof. (Additivity) Let $s \leq f, s' \leq g$ be step functions. Then, $s + s' \leq f + g$. The additivity of the integrals of step functions implies that $\int s + s' = \int s + \int s'$.

If $x \in \underline{S}(f), t \in \underline{S}(g)$, then $x + y \in \underline{S}(f + g)$. Further, $\{x + y \mid x \in \underline{S}(f), y \in \underline{S}(g)\} \subseteq \underline{S}(f + g)$. from the lemma, we have that $\sup(\{x + y \mid x \in \underline{S}(f), y \in \underline{S}(g)\}) \leq \underline{I}(f + g)$.

However, a property of sup $\implies \underline{I}(f) + \underline{I}(g) \leq \underline{I}(f + g)$.

Similarly, we get that $\overline{I}(f) + \overline{I}(g) \geq \overline{I}(f + g)$.

Since we have f, g integrable, we have that $\underline{I}(f) = \overline{I}(f), \underline{I}(g) = \overline{I}(g)$. Then, $\int f + \int g = \underline{I}(f) + \underline{I}(g) \leq \underline{I}(f + g) \leq \overline{I}(f + g) \leq \overline{I}(f) + \overline{I}(g)$. However, the last expression is equivalent to $\int f + \int g$, so equality holds for all relations. $\square$

Definition. $f : [a, b] \to \mathbb{R}$ is nonincreasing of $\forall x, y \in [a, b], x \leq y \implies f(x) \geq f(y)$.

Definition. $f : [a, b] \to \mathbb{R}$ is nondecreasing if $\forall x, y \in [a, b], x \leq y \implies f(x) \leq f(y)$.

Definition. f is monotonic if it is either nonincreasing or nondecreasing.

Lemma. *If f is monotonic on $[a, b]$, it is bounded.*

Proof. Let $C = \max[|f(a)|, |f(b)|]$.

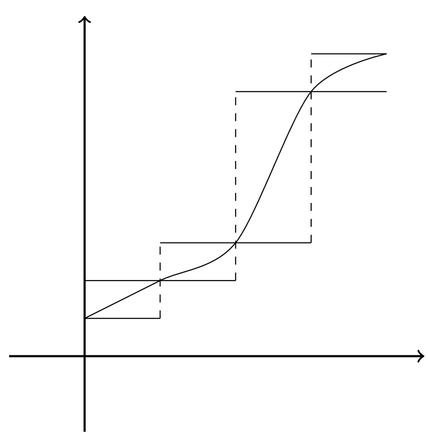
Consider the nondecreasing case first. If $x \in [a, b]$, then if $f(x) \geq 0$, then $|f(x)| = f(x) \leq f(b) = |f(b)| \leq C$, and if $f(x) \leq 0$, then $|f(x)| = -f(x) \leq -f(a) \leq |f(a)| \leq C$.

For f nonincreasing, we can consider that $-f$ is nondecreasing and the proof is then equivalent. $\square$

Theorem. *If $f : [a, b] \to \mathbb{R}$ is monotonic, then it is integrable.*

Proof. If $f : [a, b] \to \mathbb{R}$ is nondecreasing, then $-f$ is nonincreasing. By the properties of $\int$, we have that f integrable $\iff -f$ integrable.

We will show the case when f is nondecreasing.



We can also assume that $[a,b]=[0,1]$. Let $g:[0,1]\to\mathbb{R}$, with $g(x)=f(a+(b-a)x)$. By properties of translation and dilation, we have that f integrable $\iff$ g integrable. Rename g as f.

Now pick some $n\in\mathbb{Z}_{>0}$. We have $s_n,t_n:[0,1]\to\mathbb{R}$, given by

$$s_n(x)=f(\frac{\lfloor nx\rfloor}{n})$$
$$t_n(x)=\begin{cases} f(\frac{\lfloor nx\rfloor+1}{n}) & x<1 \\ f(1) & x=1\end{cases}$$

s_n,t_n are step functions as $s_n|_{(\frac{i-1}{n},\frac{i}{n})}=f(\frac{i-1}{n}), t_n|_{(\frac{i-1}{n},\frac{i}{n})}=f(\frac{i}{n})$

Further, $s_n\le f\le t_n$ as they are nondecreasing and they are constructed as the lower and upper step functions equal to f at the partitions.

Hence $\int_0^1 s_n\le \underline{I}(f)\le \overline{I}(f)\le\int_0^1 t_n$. Then we have that $0\le\overline{I}(f)-\underline{I}(f)\le\int_0^1 t_n-s_n$, from additivity of step integrals.

We will now compute $\int_0^1 t_n-s_n$.

$$\begin{aligned}\int_0^1 t_n-s_n &=\sum_{i=1}^{n}(f(\frac{i}{n})-f(\frac{i-1}{n}))\frac{1}{n}\\ &=(\frac{1}{n}(-f(0)+f(\frac{1}{n})-f(\frac{1}{n})...+f(1))\\ &=\frac{1}{n}(f(1)-f(0))\end{aligned}$$

The telescoping can be proved by induction.

Therefore, $0\le\overline{I}(f)-\underline{I}(f)\le\frac{1}{n}(f(1)-f(0)$. This holds $\forall n\in\mathbb{Z}_{>0}$.

The above implies $\forall n, n\le\frac{f(1)-f(0)}{\overline{I}(f)-\underline{I}(f)}$, which violates the Archimedian property. Thus f is integrable.

□

Definition. $f:[a.b]\to\mathbb{R}$ is piecewise monotonic if $\exists$ a partition $P=\{x_0,x_1...x_n\}$ of $[a,b]$ such that $f|_{(x_{i-1},x_i)}$ is monotonic for $1\le i\le n$.

Corollary. *If $f:[a,b]\to\mathbb{R}$ is piecewise monotonic, then f is integrable. Follows directly from results of concatenation of integrals and induction.*

Corollary. *f a linear combination of piecewise monotonic functions is itself integrable. Follows from the linearity of the integral.*

Corollary. *The function*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not a linear combination of piecewise monotonic functions.

Definition. For $x \in \mathbb{R}, n \in \mathbb{Z}_{\geq 0}$, define inductively $x^0 = 1, x^{n+1} = x^n \cdot x$.

Definition. A polynomial is a linear combination of $x \mapsto x^n$, i.e. $f(x) = \sum_{n=1}^{N} x_n c^n$.

Corollary. *(Homework) Polynomials on $[a, b]$ are integrable.*

Prop. *If $a \leq c \leq d \leq b \in \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ integrable on $[a, b]$, then $f|_{[c,d]}$ is integrable.*

Proof. Homework or Apostol. □

Definition. If $f : [a, b] \to \mathbb{R}$ is integrable, let the indefinite integral of f be the function $g(x) = \int_a^x f, (g : [a, b] \to \mathbb{R})$.

9 Limits

Definition. For $f : [a, b] \to \mathbb{R}$, $c \in [a, b], K \in \mathbb{R}$, we say that $\lim_{x\to c} = K$, whenever $\forall \epsilon > 0, \exists \delta > 0 \mid \forall x \in [a, b], 0 < |x - c| < \delta \implies |f(x) - K| < \epsilon$.

Example. 1) $f(x) = K \implies \lim_{x\to c} f(x) = K$. For any $\epsilon > 0$, pick any arbitrary δ.
2) $f(x) = x \implies \lim_{x\to c} f(x) = c$. For any $\epsilon > 0$, pick $\delta = \epsilon$.
3) $f(x) = ax, a \neq 0 \in \mathbb{R} \implies \lim_{x\to c} f(x) = ac$. For any $\epsilon > 0$, pick $\delta = \frac{\epsilon}{|a|}$.

Prop. *Consider*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

.

There is no K for which $\lim_{x\to 0} f(x) = K$. (The limit at 0 does not exist)

Proof. Suppose that it does, so $\lim_{x\to 0} f(x) = K$. Let $\epsilon = \frac{1}{2} > 0$. Then, $\exists \delta > 0 \mid 0 < |x - 0| < \delta \implies |f(x) - K| \leq \frac{1}{2}$.
Choose $x_0 \in \mathbb{Q} \mid 0 \leq |x_0| < \delta$. Choose also $x_1 \in \mathbb{R} \setminus \mathbb{Q} \mid 0 \leq |x_1| < \delta$. Then, $|f(x_0) - K| < \frac{1}{2} \implies |1 - K| < \frac{1}{2} \implies \frac{1}{2} < K < \frac{3}{2}$. Further, $|f(x_1) - K| < \frac{1}{2} \implies -\frac{1}{2} < K < \frac{1}{2}$. $\Rightarrow\!\Leftarrow$. □

Prop. $\lim_{x\to c} f(x)$ *is unique if it exists.*

Proof. Assume $\lim_{x\to c} f(x) = K_1$, $\lim_{x\to c} f(x) = K_2$.
Then, $\forall\epsilon > 0, \exists\delta_1 > 0 \mid 0 < |x - c| < d_1 \implies |f(x) - K_1| < \epsilon, \exists\delta_2 > 0 \mid 0 < |x - c| < d_2 \implies |f(x) - K_2| < \epsilon$.
Consider $\delta = \min(\delta_1, \delta_2)$, and choose x with $0 < |x - c| < \delta$. Then $|K_1 - K_2| = |K_1 - f(x) + f(x) - K_2| \leq |K_1 - f(x)| + |K_2 - f(x)| < 2\epsilon$. Thus, $K_1 = K_2$. □

Theorem. *If* $\lim_{x\to c} f(x) = K, \lim_{x\to c} g(x) = L$, *then* $\lim_{x\to c} f(x)+g(x) = K+L$. *Similarly,* $\lim_{x\to c} f(x)g(x) = KL$.
Combining, we have

$$\lim_{x\to c} \sum_{i=1}^{N} f_i(x) = \sum_{i=1}^{N} \lim_{x\to c} f_i(x)$$

.

Proof. Starting with addition, or $\epsilon > 0, \exists\delta_1, \delta_2 \mid 0 < |x - c| < \delta_1 \implies |f(x) - K| < \frac{\epsilon}{2}, 0 < |x - c| < \delta_2 \implies |g(x) - L| < \frac{\epsilon}{2}$.
Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x-c| < \delta \implies |f(x)+g(x)-K-L| \leq |f(x)-K|+|g(x)-L| < \epsilon$.
Multiplication is harder: assume that $L = 0$. Let $D = \max(1, |K|)$. We know that $\forall\epsilon_1 > 0, \exists\delta_1 > 0 \mid 0 < |x - c| < \delta_1 \implies |f(x) - K| < \epsilon_1$.
In particular, this is true for $\epsilon_1 = D$. Then $0 < |x - c| < \delta_1 \implies |f(x)| = |f(x) - K + K| \leq |f(x) - K| + |K| \leq 2D$.
We also know that $\forall\epsilon_2 > 0, \exists\delta_2 > 0 \mid 0 < |x - c| < \delta_2 \implies |g(x)| < \epsilon_2|$. In particular, for any $\epsilon > 0$, then we pick $\epsilon_2 = \frac{\epsilon}{2D}$. Then, we take $\delta = \min(\delta_1, \delta_2)$, so that $0 < |x - c| < \delta \implies |f(x)g(x) - KL| = |f(x)g(x)| < \epsilon$.
In general, we will show that $\lim_{x\to c}(f(x)g(x) - KL) = 0$. However, we can write this as $f(x)(g(x) - L) + (f(x) - K)L$. The additivity of the limit yields that the above evaluates the limit out to 0.
More specifically, we have that

$$\begin{aligned}
\lim_{x\to c}(f(x)g(x) - KL) &= \lim_{x\to c}(f(x)(g(x) - L) + (f(x) - K)L) \\
&= \lim_{x\to c}(f(x)(g(x) - L)) + \lim_{x\to c}((f(x) - K)L) \\
&= 0 + 0 = 0
\end{aligned}$$

Finally, we have that $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x)g(x) - lim_{x\to c}KL + KL = lim_{x\to c}(f(x)g(x) - KL) + KL = KL$. □

Claim. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{K}{L}$ *if* $g(x) \neq 0$, $L \neq 0$.

Proof. We know that $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} f(x) \cdot \lim_{x\to c} \frac{1}{g(x)} = K \lim_{x\to c} \frac{1}{g(x)}$.

Consider now $|\frac{1}{g(x)} - \frac{1}{L}| = |\frac{L-g(x)}{g(x)L}|$. Note that we can force $|g(x) - L| < \frac{|L|}{2}$, as we can make $|g(x) - L|$ as small as we want.

$$\begin{aligned} |L| &= |L - g(x) + g(x)| \\ &\leq |L - g(x)| + |g(x)| \\ &< \frac{|L|}{2} + |g(x)| \end{aligned}$$

And so $\frac{|L|}{2} < |g(x)|$.

Given $\epsilon > 0$, take $\epsilon_1 = \min(\frac{|L|}{2}, \frac{|L|^2}{2}\epsilon)$. Then, $\exists d_1 > 0 \mid 0 < |x - c| < \delta_1 \implies |g(x) - L| < \epsilon_1$. Applying the above, we have that $|g(x)| > \frac{|L|}{2}$.

Then, we have $|\frac{1}{g(x)} - \frac{1}{L}| = |\frac{L-g(x)}{g(x)L}| = \frac{|L-g(x)|}{|g(x)||L|} < \frac{2|L-g(x)|}{|L|^2} < \frac{2}{|L|^2}\frac{|L|^2}{2}\epsilon = \epsilon$. □

Theorem. *(Squeeze) If $f \leq g \leq h$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = K$, both existing, then $\lim_{x\to c} g(x) = K$.*

Proof. Given $\epsilon > 0$, $\exists \delta_1 > 0 \mid 0 < |x - c| < \delta_1 \implies |f(x) - K| < \epsilon$ and $\exists \delta_2 > 0 \mid 0 < |x - c| < \delta_2 \implies |h(x) - K| < \epsilon$.

Pick $\delta = \min(\delta_1, \delta_2)$. From above, we have that $-\epsilon + K < f(x) < \epsilon + K$, and $-\epsilon + K < h(x) < \epsilon + K$. Further, $K - \epsilon < f(x) \leq g(x) \leq h(x) < K + \epsilon \implies -\epsilon + K < g(x) < \epsilon + K \implies |g(x) - K| < \epsilon$. □

Definition. Let $f : [a, b] \to \mathbb{R}$ is continuous at $c \in [a, b]$ if $\lim_{x\to c} f(x) = f(x)$. It is discontinuous if it is not continuous, and f is continuous if it is continuous $\forall c \in [a, b]$.

Example. $f(x) = K \implies \lim_{x\to c} f(x) = K \implies f$ is continuous.

Example. $f(x) = ax \implies \lim_{x\to c} f(x) = \lim_{x\to c} ax = ac = f(x) \implies f$ is continuous.

Example. For $f(x) = ax^n$, induction and the multiplicative nature of lim yields that $\lim_{x\to c} f(x) = ac^n = f(x) \implies f$ is continuous.

Example. From the above, we have that polynomials are continuous.

Example.

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

is not continuous. We have that $\lim_{x\to 0} f(x) = 0 \neq f(0) = 1$. Thus f is not continuous at the origin.

Example.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not continuous. We show that $\lim_{x \to c} f(x)$ does not exist, and so f is everywhere discontinuous.

Theorem. *If f, g are continuous at c, then $f + g, fg$ are continuous. So are $f - g, \frac{f}{g}$, the last one if $g(c) \neq 0$.*

Proof. These follow from the previous work on limits. □

Prop. *These are easy to prove:*

1. *f is continuous at $c \iff \forall \epsilon > 0, \exists d > 0 \mid |x - c| < \delta \implies |f(x) - f(x)| < \epsilon$.*
2. *$\lim_{x \to c} f(x) = K \iff \lim_{h \to 0} f(c + h) = K$.*

Theorem. *If $f : [a, b] \to \mathbb{R}$ is continuous at $c \in [a, b]$, and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(c)$, then $g \circ f : [a, b] \to \mathbb{R}$ is also continuous at c.*

Proof. We know that $\forall \epsilon_1 > 0, \exists \delta_1 > 0 \mid |x - c| < \delta_1 \implies |f(x) - f(c)| < \epsilon_1$ and $\forall \epsilon_2 > 0, \exists \delta_2 \mid |y - f(c)| < d_2 \implies |g(y) - g(f(c))| < \epsilon_2$.

Given $\epsilon > 0$, we take $\epsilon_2 = \epsilon$, getting then a $\delta_2 > 0$ as above. We now take $\epsilon_1 = \delta_2$, getting then a $\delta_1 > 0$. We now take $\delta = \delta_1$.

We have that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon_1 = \delta_2$. Now let $y = f(x)$, so that we also have $|g(f(x)) - g(f(c))| = |(g \circ f)(x) - (g \circ f)(c)| < \epsilon_2 = \epsilon$. □

10 Consequences of Continuity

In order to prove that all continuous functions are integrable, we will first show that we have the following:

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is bounded.*

Remark. This will be different from Apostol. He assumes that it is not bounded and derives a contradiction by zeroing in on the unbounded domain. Our approach will focus on expanding on a interval in which we know the function is bounded, eventually expanding to the whole interval. This can be done via contradiction.

Lemma. *Let $S = \{x \in [a, b] \mid f$ is bounded on the interval $[a, x]\}$. Then the supremum of this set c exists, and $\exists d > c \mid f$ is bounded on $[a, \min(b, d)]$.*

Proof. (Lemma) We know that $a \in S$, as f is bounded by $f(a)$ on $[a, b]$, b is also an upper bound for S. Thus, $c = \sup(S)$ exists.

Take $\epsilon = 1$ in the definition of continuity at c, such that $\exists \delta \mid |x - c| < \delta \implies |f(x) - f(c)| < 1 \implies -1 < f(x) - f(c) < 1 \implies f(c) - 1 < f(x) < 1 + f(c)$. Then, let $K = \max(|f(c) - 1|, |f(c) + 1|)$ is a bound on f.

Hence, f is bounded on $(c - \delta, c + \delta)$. By the approximation property of sup, $\exists y \in S$ such that $c - \delta < y \leq c$. Then, f is bounded on $[a, y]$. Thus, f must be bounded on $[a, c + \delta)$.

Let $d = c + \frac{\delta}{2}$, and now f is bounded on $[a, d]$. □

Proof. The theorem follows from the lemma quite easily. Let c, d be as in the lemma, and we have that $S \subseteq [a, b] \implies c \leq b$. If $c < b$, then $c < \min(b, d)$, but the lemma yields that f is bounded on $[a, \min(b, c)]$. However, $\min(b, c) > \sup(S)$ but $\min(b, c) \in S, \Rightarrow\Leftarrow$.

Thus, we have that $c = b$, so $\min(b, d) = b$, meaning that the lemma has that f is bounded on $[a, b]$. □

Definition. Let $f : [a, b] \to \mathbb{R}$. The absolute minimum of f, if it exists, is the value K such that $\forall x \in [a, b], f(x) \geq K$ and $\exists c \in [a, b]$ such that $f(c) = K$.

Theorem. *(Extreme Value Theorem) A continuous function $f : [a, b] \to \mathbb{R}$ has an absolute min/max.*

Proof. Let $M(f) = \sup\{f(x) \mid x \in [a, b]\}, m(f) = \inf\{f(x) \mid x \in [a, b]\}$.

Note that existence follows from the previous theorem. Suppose that there is no $c \in [a, b] \mid f(c) = M(f)$. Then the following function $M(f) - f(x)$ is never zero; particularly, we can see that $g(x) = \frac{1}{M(f) - f(x)}$ exists on $[a, b]$ and is continuous everywhere.

This function is bounded by the previous function; say that $|g(x)| \leq K$. However, the reciprocal property of sup $\implies \exists x \in [a, b] \mid 0 \leq M(f) - f(x) < \frac{1}{K}$.

Then, $|g(x)| = \frac{1}{|M(f) - f(x)|} > K$, $\Rightarrow\Leftarrow$.

The proof is similar for minima. □

Example.

Theorem. *Let $f : [a, b] \to [a, b]$ be continuous and satisfying $x \neq y \implies |f(x) - f(y)| < |x - y|$ (f is contracting). Then $\exists! c \in [a, b] \mid f(c) = c$. This is called a fixed point.*

Proof. We can easily show uniqueness, as if $f(c) = c, f(d) = d \implies |f(c) - f(d)| < |c - d|$, so $|c - d| = 0$.

For existence, let $g(x) = |f(x) - x|$. then g is continuous from limit rules and the properties of the absolute value. The Extreme Value Theorem then yields that $\exists c \mid \forall x \in [a, b], g(c) \leq g(x)$. Suppose that $g(c) \neq 0$, so $f(c) \neq c$, and $|g(f(c))| = |f(f(c)) - f(c)| < |f(c) - c| = g(c)$, and so $g(f(c)) < g(c)$. $\Rightarrow\Leftarrow$

Thus, we have that $g(c) = 0 \implies f(c) = c$. □

Definition. Let $f : [a,b] \to \mathbb{R}$, let $\text{span}(f) = M(f) - m(f)$, if the latter half exists.

Theorem. *(Small Span Theorem) If $f : [a,b] \to \mathbb{R}$ is continuous, then $\forall \epsilon > 0, \exists$ a partition $\{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that $\text{span}(f|_{[x_i, x_{i+1}]}) < \epsilon$.*

Lemma. *Fix $\epsilon > 0$. Let $S = \{x |$ the above theorem holds for $f_{[a,x]}\}$. Then $\exists c = \sup(S)$, and $\exists d > c$ such that the theorem is true on $[a, \min(b,d)]$.*

Proof. (Lemma) $a \in S$, S is bounded above by b, so $c = \sup(S)$ exists. We apply continuity at c such that $\exists \delta > 0 \mid |x - c| < \delta \implies |f(x) - f(c)| < \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} < f(x) - f(c) < \frac{\epsilon}{2}$.
The extreme value theorem applied to $f|_{[c - \frac{\delta}{2}, c + \frac{\delta}{2}] \cap [a,b]}$ yields a maximum M and a minimum m. Then, we have that $M < f(c) + \frac{\epsilon}{2}, m > f(c) - \frac{\epsilon}{2}$. Then, $\text{span}(f|_{[c - \frac{\delta}{2}, c + \frac{\delta}{2}] \cap [a,b]}) < \epsilon$.
Then, pick $d = c + \frac{\delta}{2}$. The approximation property then gives $y \in S \mid y > c - \frac{\delta}{2}$. Then, $\exists P = \{x_0, ..., x_n\}$ such that the partition holds $\implies P' = \{x_0, ..., x_n = y, \min(b,d)\}$ is a partition that witnesses the lemma. ☐

Proof. The theorem follows from the lemma the same way as the theorem on boundedness followed from its own lemma. ☐

Theorem. *$f : [a,b] \to \mathbb{R}$ is continuous $\implies f$ is integrable.*

Proof. Since f is continuous, then f must be bounded. Given $\epsilon > 0$, the Small Span Theorem yields $P = \{x_0, ..., x_n\} \mid \text{span}(f|_{[x_{i-1}, x_i]}) < \frac{\epsilon}{b-a} \forall i, 1 \leq i \leq n$.
Let

$$s(x) = \begin{cases} m(f|_{[x_{i-1}, x_i]}) & x \in [x_{i-1}, x) \\ f(b) & x = b \end{cases}$$

$$t(x) = \begin{cases} M(f|_{[x_{i-1}, x_i]}) & x \in [x_{i-1}, x) \\ f(b) & x = b \end{cases}$$

Then $s \leq f \leq t$, and

$$\begin{aligned}
\int_a^b (t-s) &= \sum_{i=1}^{n} (M(f|_{[x_{i-1},x_i]}) - m(f|_{[x_{i-1},x_i]}))(x_i - x_{i-1}) \\
&= \sum_{i=1}^{m} \operatorname{span}(f|_{[x_{i-1},x_i]})(x_i - x_{i-1}) \\
&< \sum_{i=1}^{m} \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a} \sum_{i=1}^{m} (x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a}(b-a) = \epsilon
\end{aligned}$$

This is equivalent to f being integrable by a homework problem. □

Theorem. *If $f : [a,b] \to \mathbb{R}$ is integrable, then $g(x) = \int_a^x f, (g : [a,b] \to \mathbb{R})$ is continuous everywhere.*

Proof. Since f is integrable, f must be bounded; say $\forall x \in [a,b], |f(x)| \leq K$. Now, for any $\epsilon > 0$, take $\delta = \frac{\epsilon}{K}$. Then $0 < |y-x| < \delta \implies |g(y) - g(x)| = |\int_a^y f - \int_a^x f| = |\int_x^y f| \leq \int_x^y |f| \leq \int_x^y K = (y-x)K < \delta K < \epsilon$. □

Theorem. *(Bolzano) Let $f : [a,b] \to \mathbb{R}$ be continuous and $f(a) < 0 < f(b)$. Then, $\exists x \in [a,b] \mid f(x) = 0$.*

Remark. This is false if we have a single discontinuity, or if $\mathbb{R}$ is replaced by $\mathbb{Q}$. For example, take $a = 0, = 2, f(x) = x^2 - 2, f : \mathbb{Q} \to \mathbb{Q}$. This is perfectly continuous, but never 0 anywhere.

Lemma. *If $f : [a,b] \to \mathbb{R}$ is continuous and $f(x) > 0$, then $\exists \delta > 0 \mid 0 < |y-x| < \delta \implies f(y) > 0$.*

Proof. (Lemma) In the definition, take $\epsilon = f(x) > 0$. Continuity at $x \implies \exists \delta > 0 \mid 0 < |y-x| < \delta \implies |f(x) - f(y)| < \epsilon = f(x) \implies -f(x) < -f(x) + f(y) < f(x) \implies 0 < f(y)$. □

Proof. (Bolzano) Let $S = \{x \in [a,b] \mid f(x) < 0\}$. We have that $a \in S$, and as $f(b) > 0$, S is bounded above by b. Thus, $c = \sup(S)$ exists.

Assume that $f(c) > 0$, so that $c > a$. Applying the lemma to f at c, $\exists \delta > 0 \mid |y - c| < \delta \implies f(y) > 0$. However, $\exists y < c$ with $f(y) > 0$, but this implies that $y < c$ is an upper bound for S. $\Rightarrow\Leftarrow$

Assume that $f(c) < 0$, so that $c < b$. Applying the lemma to $-f$ at c, $\exists \delta > 0 \mid 0 < |y-c| < \delta \implies f(y) < 0$. In particular, $\exists y > c \mid f(y) < 0 \implies y \in S$. $\Rightarrow\Leftarrow$

Conclude that $f(c) = 0$. □

Theorem. *(Intermediate Value Theorem) Suppose* $g : [a, b] \to \mathbb{R}$ *is continuous,* $g(a) < k < g(b)$*. Then* $\exists c \in [a, b] \mid g(c) = k$*.*

Proof. Apply Bolzano to $f(x) = g(x) - k$. □

11 Forming Inverses

Prop. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous and strictly increasing. Then* f *induces a bijection* $f : [a, b] \to [c, d]$ *where* $c = f(a), d = f(b)$*, and the inverse function* $f^{-1} : [c, d] \to [a, b]$ *is continuous and strictly increasing.*

Proof. The full proof is in Apostol. Injectivity is a result of f being strictly increasing. Surjectivity follows from the Intermediate Value Theorem. □

Example. We form $x^{\frac{1}{n}} = f^{-1}(x)$ by taking $f(x) = x^n, f : [0, b] \to \mathbb{R}$.

12 Derivatives

Definition. Given $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$, we say that f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists. If so, the derivative $f'(x)$ is the value. f is differentiable if it is differentiable at all such x.

Example.

$$f(x) = C \implies \lim_{h \to 0} \frac{C - C}{h} = 0$$

Example.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

is differentiable.

Theorem. f *is differentiable* $\implies$ f *is continuous.*

Proof. Remember that $lim_{h \to 0}(f(x + h) - f(x)) = 0 \iff f$ is continuous at x. Further, we have that

$$\begin{aligned} \lim_{h \to 0}(f(x + h) - f(x)) &= (\lim_{h \to 0} h)(\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}) \\ &= 0 \cdot f'(x) \\ &= 0 \end{aligned}$$

$\square$

Theorem. *Suppose $f : [a,b] \to \mathbb{R}, g : [a,b] \to \mathbb{R}$ are differentiable at x. Then the following are also differentiable at x and take the respective values:*

1. $(f+g)' = f' + g'$
2. $(f-g)' = f' - g'$
3. $(fg)' = fg' + f'g$
4. $\frac{f}{g} = \frac{fg' - f'g}{g^2}$ (*if* $g(x) \neq 0$)

Proof. For 1, 2,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= f'(x) + g'(x)\end{aligned}$$

The statement follows from well-definedness result for lim.
For 3,

$$\begin{aligned}(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h} \\ &= (\lim_{h\to 0} g(x+h))(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}) + (\lim_{h\to 0} f(x))(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}) \\ &= gf' + g'f\end{aligned}$$

The statement follows from well-definedness result for lim.
For 4, it is sufficient to compute $(\frac{1}{g})'$.

$$\begin{aligned}(\frac{1}{g})'(x) &= \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\ &= \lim_{h\to 0} \frac{g(x) - g(x+h)}{hg(x)g(x+h)} \\ &= (\lim_{h\to 0} \frac{g(x)-g(x+h)}{h})(\lim_{h\to 0} \frac{1}{g(x)g(x+h)}) \\ &= -\frac{g'}{g^2}\end{aligned}$$

We can assume $g(x+h) \neq 0$ as g is continuous and therefore $\exists$ neighborhood of x with $g(y) \neq 0 \forall |x-y| < \delta$.

Combining 3 and the above, we have:

$$\begin{aligned}(\frac{f}{g})' &= f(\frac{1}{g})' + f'\frac{1}{g} \\ &= -\frac{fg'}{g^2} + f'\frac{1}{g} \\ &= \frac{f}{g} = \frac{fg' - f'g}{g^2}\end{aligned}$$

□

Corollary. *We have now several properties, such as that $(cg)' = cg'$, which can be extended to show that the derivative is linear.*

Corollary.

$$(\sum_{n=0}^{N} x^n)' = \sum_{n=1}^{N} na_x x^{n-1}$$

Proof. We have from linearity and induction that we can work termwise, such that

$$(\sum_{n=0}^{N} a_n x^n)' = \sum_{n=0}^{N} a_n (x^n)'$$

To compute x^n, we have that $(x^0)' = 0, (x^1)' = 1$. Taking the inductive step, we will see that $(x^n)' = nx^{n-1}$ if $x \geq 1$, and 0 if $x = 0$. Assuming for n,

$$(x^{n+1})' = (x^n \cdot x)' = x^n \cdot 1 + nx^{n-1} \cdot x = (n+1)x^n$$

□

Theorem. *(Chain Rule) Let $f : [a,b] \to [c,d], g : [c,d] \to \mathbb{R}$. Assume that f is differentiable at x, and g is differentiable at $f(x)$. Then, $g \circ f$ is differentiable at x and $(g \circ f)'(x) = f'(x) \cdot g'(f(x))$.*

Proof. (Fake) Let $k = f(x+h) - f(x)$.

$$\begin{aligned}\lim_{h \to 0} \frac{(g \circ f)(x+h) - (g \circ f)(x)}{h} &= \lim_{h \to 0} \frac{(g \circ f)(x+h) - (g \circ f)(x)}{f(x+h) - f(x)} \cdot \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{g(f(x)+k) - g(f(x))}{k} f'(x)\end{aligned}$$

Further, as $h \to 0$, $k \to 0$, so the final expression is $g'(f(x)) \cdot f'(x)$.

However, we have no guarantee that $k \neq 0$, so this doesn't work. □

Proof. Regard x as a constant and define the following functions:

$$F(h) = f(x+h) - f(x)$$
$$G(k) = \begin{cases} \frac{g(f(x)+k)-g(f(x))}{k} & k \neq 0 \\ g'(f(x)) & k = 0 \end{cases}$$

Now, we know that F, G are both continuous. For $h \neq 0$, we have that $\frac{(g \circ f)(x+h)-(g \circ f)(x)}{h} = (G \circ F)(h) \cdot \frac{f(x+h)-f(x)}{h}$.
Now, if $F(h) \neq 0$, then this follows the same was as in the fake proof. If $F(h) = 0$, then we have that $f(x) = f(x+h) \implies g(f(x+h)) - g(f(x)) = 0 \implies G(0) = 0$.
Taking

$$\lim_{h \to 0} LHS = (g \circ f)'(x)$$
$$\lim_{h \to 0} RHS = \lim_{h \to 0} (G \circ F)(h) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = g'(f(x)) \cdot f'(x)$$

□

13 Fundamental Theorem of Calculus

Theorem. *(Fundamental Theorem of Calculus I) Let $f : [a,b] \to \mathbb{R}$ be integrable, pick $c \in [a,b]$, let $g : [a,b] \to \mathbb{R}$ be given by $g(x) = \int_c^x f$. If f is continuous at some point x, then g is differentiable at x and $g'(x) = f(x)$.*

Proof.

$$\begin{aligned}
\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} &= \lim_{h \to 0} \frac{\int_c^{x+h} f - \int_c^x f}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} (f(t) - f(x))dt + \frac{1}{h} \int_x^{x+h} f(x)dt \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} (f(t) - f(x))dt + f(x)
\end{aligned}$$

Thus, it will suffice to show that $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} (f(t) - f(x))dt = 0$. From the continuity of f, we have that $\forall \epsilon > 0, \exists \delta \mid |t - x| < \delta \implies |f(t) - f(x)| < \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} < f(t) - f(x) < \frac{\epsilon}{2}$.

Then, if $h \in (0, \delta)$, then $\int_x^{x+h} -\frac{\epsilon}{2} dt \leq \int_x^{x+h} (f(t) - f(x)) dt \leq \int_x^{x+h} \frac{\epsilon}{2} dt \implies |\int_x^{x+h} (f(t) - f(x)) dt| \leq \frac{|h|\epsilon}{2}$.
Then, if $h \in (-\delta, 0)$, then $\int_{x+h}^{x} -\frac{\epsilon}{2} dt \leq \int_{x+h}^{x} (f(t) - f(x)) dt \leq \int_{x+h}^{x} \frac{\epsilon}{2} dt \implies |\int_{x+h}^{x} (f(t) - f(x)) dt| = |\int_x^{x+h} (f(t) - f(x)) dt| \leq \frac{|h|\epsilon}{2}$.
Hence, $0 < |h| < \delta \implies \left| \frac{\int_x^{x+h} (f(t) - f(x)) dt}{h} \right| = \frac{|\int_x^{x+h} (f(t) - f(x)) dt|}{h} \leq \frac{\epsilon |h|}{2} \frac{1}{|h|} < \epsilon$. □

Example.

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases}$$

Then, $g(x) = \int_0^x f(t) dt = -|x|$, not differentiable at 0.

Example.

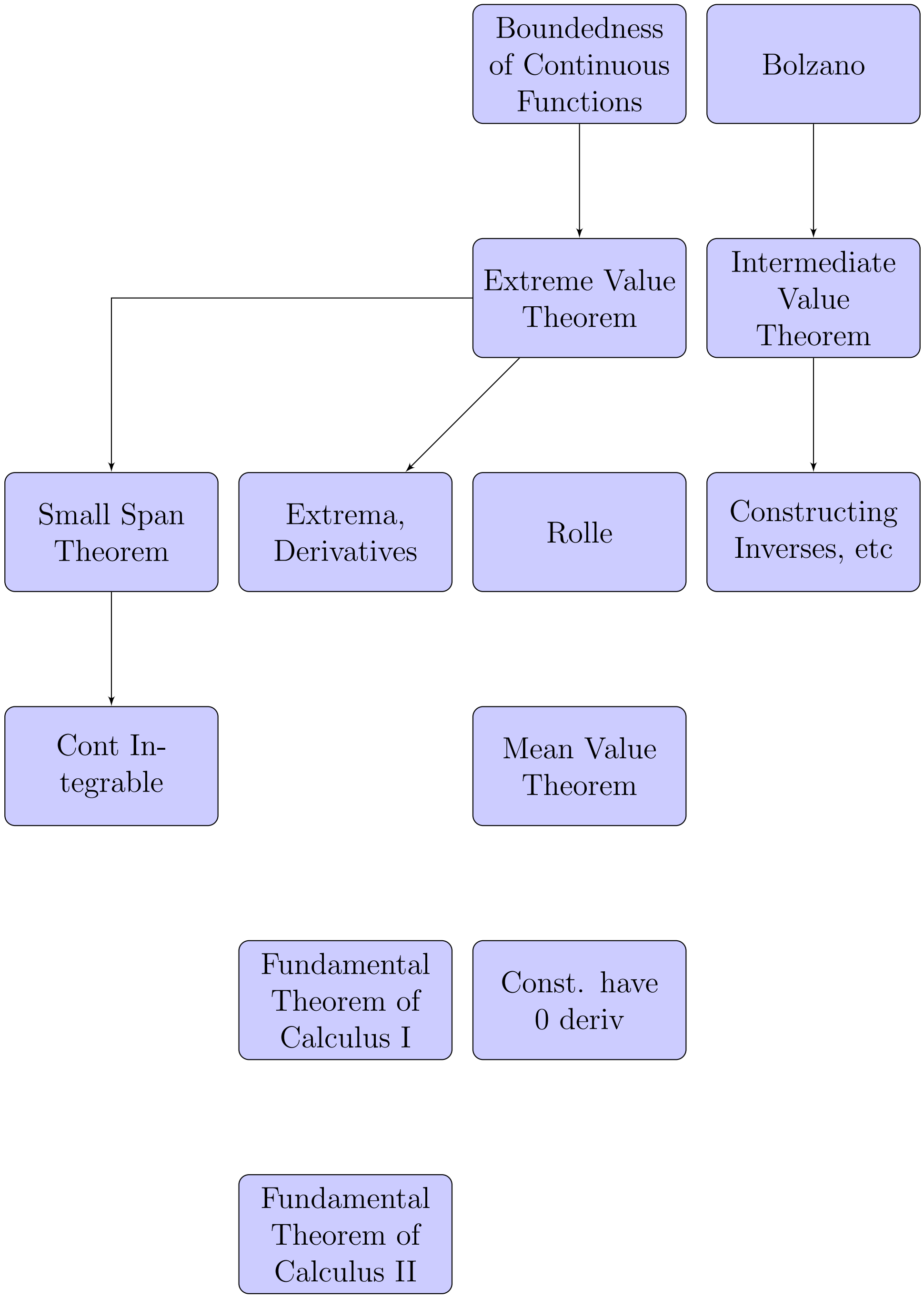
$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then, $g(x) = \int_0^x f(t) dt = x$, which has derivative $1 \neq f(0)$ at 0.

Theorem. *(Fundamental Theorem of Calculus II) If $g : [a, b] \to \mathbb{R}$ satisifying $g' = f$ is a continuous function on $[a, b]$, then $\forall x, y \in [a, b], g(y) - g(x) = \int_x^y f$.*

Proof. Fix $x \in [a, b]$ and define $h(y) = g(y) - g(x) - \int_x^y f$. The Fundamental Theorem of Calculus I yields that $h'(y) = g'(y) - f(y) = f(y) - f(y) = 0$. However, only constant functions have derivative 0, and we have that $h(x) = 0$, so $h(y) = 0$ and the theorem follows. □

Remark. We have mappings between continuous functions and functions with continuous derivatives. From continuous functions to those with continuous derivatives, we have that this mapping is called $I_c(f) = (x \to \int_c^x f)$, and the opposite mapping is the differential operator, $D(f) = f'$. The Fundamental Theorem of Calculus tells us that $D \circ I_c = \mathrm{id}, I_C \circ D(f) = f + C_f$.



14 Computing Integrals

Definition. g is an antiderivative of f if $g' = f$.

Theorem. *If* $u : [a, b] \to \mathbb{R}$ *is differentiable with* $u' : (a, b) \to \mathbb{R}$ *continuous and* $f : \mathbb{R} \to \mathbb{R}$

continuous, then

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(t)dt$$

Proof. Let $c \in \mathbb{R}$, $g(x) = \int_c^x f(t)dt$. Then, $(g \circ u)'(x) = g'(u(x))u'(x)$, which is differentiable by the Fundamental Theorem of Calculus.

Then,

$$\begin{aligned}
\int_{U(a)}^{u(b)} f(t)dt &= \int_c^{u(b)} f(t)dt - \int_c^{u(a)} f(t)dt \\
&= g(u(b)) - g(u(a)) \\
&= (g \circ u)(b) - (g \circ u)(a) \\
&= \int_a^b (g \circ u) = \int_a^b f(u(x))u'(x)dx
\end{aligned}$$

□

Prop.

$$\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1} \quad n \neq -1, n \in \mathbb{Q}$$

Proof. Check that $g(x) = \frac{x^{n+1}}{n+1}$ is an antiderivative of x^n. Apply Fundamental Theorem of Calculus II. □

15 Defining Useful Functions

Definition.

$$\log : \mathbb{R}_{>0} \to \mathbb{R}, \log(x) = \int_1^x \frac{1}{t} dt$$

Prop. log *is strictly increasing.*

Proof. Fundamental Theorem of Calculus implies that $\log'(x) = \frac{1}{x} > 0 (x > 0)$. A corollary of Mean Value Theorem says that log is strictly increasing. □

Remark.

$$\log(1) = \int_1^1 \frac{dt}{t} = 0$$

Prop.

$$\forall x \in \mathbb{R}_{>0}, n \in \mathbb{Z}, \log(x^n) = n \log(x)$$

Proof. This is easily checked for $n = 0, n = 1$. For $n > 1$, we induct on n using the following property; same with $n < 0$. ☐

Prop.

$$\forall x, y \in \mathbb{R}_{>0}, \log(xy) = \log(x) + \log(y)$$

Proof.

$$\begin{aligned}\log(xy) &= \int_1^x \frac{1}{t}dt + \int_x^{xy} \frac{1}{t}dt \\ &= \log(x) + x\int_1^y \frac{1}{tx}st = \log(x) + \log(y)\end{aligned}$$

☐

Prop. log *is unbounded above and below.*

Proof. Pick $c \in \mathbb{R}$. The archimedian property of the reals $\implies \exists n \in \mathbb{Z} \mid n > \frac{C}{\log(2)}$. Then, consider $\log(2^n) = n\log(2) > C$. The lower bound can be checked by considering $\log(2^{-n})$. ☐

Remark. log is a bijection taking $\mathbb{R}_{>0} \to \mathbb{R}$, as it is strictly increasing (injective) and unbounded below and above and continuous (surjective).

Definition. Let $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ be the inverse function of log.

Remark. The properties of inverse functions yield that *exp* is continuous.

Definition.

$$e := \exp(1)$$

Prop. $\forall x, y \in \mathbb{R}$, $\exp(x+y) = \exp(x)\exp(y)$.

Proof.

$$\begin{aligned}\log(\exp(x+y)) &= x + y \\ &= \log(\exp(x)) + \log(\exp(y)) \\ &= \log(\exp(x)\exp(y)) \\ \implies \exp(x+y) &= \exp(x)\exp(y)\end{aligned}$$

☐

Prop. $\forall x \in \mathbb{Q}, \exp(x) = e^x$

Proof. Use induction with the base case $x = 1$ and the above proposition to show it for $x \in \mathbb{Z}$. Continue with the extension to the rationals by writing it as a quotient. ☐

Definition. $\forall x \in \mathbb{R}, e^x = \exp(x)$. Then, for

$$a \in \mathbb{R}_{>0}, x \in \mathbb{R}, a^x := \exp(x \log(a))$$

Prop. $\exp'(x) = \exp(x)$.

Proof.

$$\begin{aligned} \lim_{h \to 0} \frac{\exp(x+h) - \exp(x)}{h} &= \lim_{h \to 0} \frac{\exp(x)\exp(h) - \exp(x)}{h} \\ &= \lim_{h \to 0} \exp(x) \frac{\exp(h) - 1}{h} \end{aligned}$$

Now, let $k = e^h - 1$, is $h = \log(k+1)$. As $h \to 0, k \to 0$. Likewise, as $k \to 0, h \to 0$. Then,

$$\lim_{k \to 0} \frac{k}{\log(k+1)} = \lim_{k \to 0} \left(\frac{\log(k+1) - \log(0)}{k} \right)^{-1} = (\log'(1))^{-1} = 1$$

Thus, we have that

$$\lim_{h \to 0} \exp(x) \frac{\exp(h) - 1}{h} = \exp(x)$$

□

16 Sequences

Definition. A sequence of real numbers is a function $a_n : \mathbb{Z}_{>0} \to \mathbb{R}$ (or $\mathbb{Z}_{\geq 0}$), and we will write $a_i = a(i)$. The entire sequence is $\{a_n\}$.

Definition. A sequence $\{a_n\}$ has limit L if for all $\epsilon > 0, \exists N \in \mathbb{Z}_{>0} \mid \forall n \geq N, |a_n - L| < \epsilon$. This limit L is denoted as $\lim_{n \to \infty} a_n$.

Example.

$$\lim_{n \to \infty} \frac{1}{e^n} = 0$$

To see this, pick $N = \log(1 + \frac{1}{\epsilon})$.

Remark. Limits of sequences satisfy the same limit laws as for functions (e.g. $\lim_{n \to \infty}(a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$, etc).

Further, these limits are unique, and there is an analogous squeeze theorem.

Definition. $\{a\}$ is increasing if $\forall m, b \in \mathbb{Z}_{>0}, m \leq n \implies a_m \leq a_n$.

Theorem. *(Monotone Sequence Theorem) If $\{a\}$ is non-decreasing and $\exists c \in \mathbb{R} \mid a_n \leq c \forall n$, then the limit exists and is $\leq c$.*

Proof. Consider the set $S = \{a_n \mid n \in \mathbb{Z}_{>0}\} \subseteq \mathbb{R}$. S is nonempty and bounded above, so the supremum exists. Let $L = \sup(A)$. Given $\epsilon > 0$, the approximation property with ϵ implies that $\exists N \in \mathbb{Z}_{>0}$ such that $|L - a_n| = L - a_n < \epsilon$. Non-decreasingness yields that $\forall n \geq N$, $a_N \leq a_n \implies L - a_n < \epsilon$. □

Prop. *If $\{a_n\}$ is nondecreasing and convergent to L, then $\forall n \in \mathbb{Z}_{>0}, a_n \leq L$.*

Proof. If not, say that $\exists m \in \mathbb{Z}_{>0}$ with $a_m > L$. Let $\epsilon = a_m - L > 0$. Then, we have that $\forall n \geq m$, $a_m \leq a_n \implies a_n - L = |L - a_n| > \epsilon$, which violates the $\epsilon - \delta$ definition of the limit. □

Prop. *(Limit definition of e)* $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.

Proof. Choose n. For $t \in [1, 1 + \frac{1}{n}]$, we have that $\frac{n}{n+1} \leq \frac{1}{t} \leq 1 \implies \int_1^{1+\frac{1}{n}} \frac{n}{n+1} dt \leq \int_1^{1+\frac{1}{n}} \frac{dt}{t} \leq \int_1^{1+\frac{1}{n}} dt \implies \frac{1}{n+1} \leq \log(1+\frac{1}{n}) \leq 1+\frac{1}{n} \implies e^{\frac{1}{n+1}} \leq 1+\frac{1}{n} \leq e^{\frac{1}{n}} \implies e \leq (1+\frac{1}{n})^{\frac{1}{n+1}}, 1+\frac{1}{n}^n \leq e \implies \frac{e}{1+\frac{1}{n}} \leq (1+\frac{1}{n})^n \leq e$. The corresponding squeeze theorem yields what we wanted. □

17 Series

Definition. If $\{a_n\}_{n=0}$ is a sequence, let $b_m = \sum_{n=0}^m a_n$ be the sequence of partial sums of a_n (or just the series of) a_n.

Its limit $\lim_{m\to\infty} b_m$ is the sum of the series, if it exists. If the limit exists, call this series convergent and divergent otherwise, and it will be denoted

$$\sum_{n=0}^{\infty} a_n$$

Remark. We have the following statements about series:

1. For $k \in \mathbb{Z}_{>0}$, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=k}^{\infty} a_n := \sum_{n=0}^{\infty} a_{n+k}$ either both converge or both diverge.
2. If $\{a_n\}$ and $\{b_n\}$ are eventually equal (i.e. $\exists N$ such that for $n \geq N$, $a_n = b_n$) then they either both converge or diverge.
3. If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Prop. *(Telescoping sums) Suppose that $\{a_n\}$ is a sequence such that $a_n = b_n - b_{n-1}$ for some other sequence $\{b_n\}$. Then, $\lim_{n\to\infty} b_n$ exists if and only if $\sum_{n=0}^{\infty} a_n$ converges.*

Proof. Partial sums $\sum_{n=0}^{m} a_n = \sum_{n=0}^{m} b_n - b_{n-1} = b_m - b_{-1}$. Taking the limits, we see that $\sum_{n=0}^{\infty} a_n = \lim_{m\to\infty} b_m - b_{-1}$. Thus, they have the same existence status and can be computed in terms of each other. □

Example.

$$\sum_{n=1}^{\infty} \log(\frac{n}{n+1}) = \sum_{n=1}^{\infty} [\log(n) - \log(n+1)]$$

but $\lim_{n\to\infty} \log(n)$ diverges, so the sum diverges.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\sum_{n=1}^{\infty}(\frac{1}{2n-1} - \frac{1}{2n+1})$$

We have that the sum is $\frac{1}{2}\lim_{n\to\infty} b_n + 1 = \frac{1}{2}$.

Prop. *(Geometric series) Consider $\sum_{n=0}^{\infty} x^n$. Supposing that $x \neq 1$, we have that $b_m = \sum_{n=0}^{m} a_n$ has the property that $(1-x)b_n = 1 - x^n \implies b_n = \frac{1-x^n}{1-x}$. When $x < 1$, then the sum converges, or else it converges to $\frac{1}{1-x}$.*

Theorem. *(Integral Test) Let $f : \mathbb{R}_{>1} \to \mathbb{R}$ be nonnegative and monotonically decreasing. Let $s_n = \sum_{i=1}^{n} f(i), t_n = \int_1^n f$. Then these either both converge or both diverge.*

Proof. Let $g(x) = f(\lfloor x \rfloor)$. Because g is decreasing, we have that $g \geq f \implies \int_1^{n+1} g \geq \int_1^{n+1} f \implies s_n \geq t_n$.

Now let $h(x) = f(\lceil x \rceil)$. Then, $h \leq f$ on the same interval as before, $[1, n+1]$. We integrate both sides and arrive at $\int_1^{n+1} h \leq \int_1^{n+1} f \implies s_n - (f_1 - f_{n+1}) \leq t_n$.

Then, if t_n converges, then t_n is bounded, so $s_n - (f(1) - f(n))$ is bounded, so s_n is bounded and thus converges as it is monotonic. Conversely, if s_n converges, then t_n must also be bounded, and so converges. □

Remark. Note that this guarantees that the eventual limits of the function only differ by at most $f(1)$.

Example. The above shows that the harmonic series diverges. Further, we have that $\lim_{n\to\infty}(\sum_{i=1}^{n} \frac{1}{i} - \log(n+1))$ is bounded. In fact it is γ, the Euler-Mascheroni constant.

Example. Similarly, we have that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$.

Theorem. *Assume that $a_n \geq 0$. Then, $\sum_{i=1}^{\infty} a_i$ converges $\iff$ the sequence of partial sums is bounded.*

Proof. Since we have a_n nonnegative, we have that $\sum_{i=1}^{\infty} a_i$ is monotonic. The result follows from the Monotone Convergence Theorem. □

Corollary. *(Comparison test) Assume $a_n \geq 0$, $b_n \geq 0$, and that $\exists c \in \mathbb{R}$ such that $\forall n, a_n \leq cb_n$. Then if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{i=1}^{\infty} a_n$.*

Proof.

$$s_n = \sum_{i=1}^{n} a_i$$
$$t_n = \sum_{i=1}^{n} b_i$$
$$\implies s_n \leq ct_n$$

Then, since ct_n is bounded, s_n is bounded, and so $\sum_{n=1}^{\infty} a_n$ converges. □

Prop. *(Limit comparison test) Assume that $a_n > 0, b_n \geq 0$, and that $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$. Then $\sum a_i$ converges $\iff$ $\sum b_i$ converges.*

Proof. For $\epsilon = \frac{1}{2}$, $\exists N \mid n > N \implies \frac{1}{2} \leq \frac{b_b}{a_n} \leq \frac{3}{2} \implies a_n \leq 2b_n, b_n \leq \frac{3}{2}a_n$. Thus, since the first N terms don't matter for convergence, we have that either one converging implies the other converges. □

Definition. $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Prop. *Absolute converges $\implies$ convergence, and in this case*

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} a_n$$

Proof. Suppose that $\sum |a_n|$ converges. Let $b_n = a_n + |a_n|$. We have that b_n is nonnegative and also $b_n \leq 2|a_n|$. Since we have that $\sum |a_n|$ converges, b_n converges. Further, since

$$\sum b_n - \sum |a_n| = \sum (b_n - |a_n|) = \sum a_n$$

then $\sum a_n$ must converge.
Then,

$$\begin{aligned} -\sum |a_n| = \sum -|a_n| &\leq \sum b_n - |a_n| \\ &= \sum b_n - \sum |a_n| \; (= \sum a_n) \\ &\leq 2|a_n| - \sum |a_n| \\ &= \sum |a_n| \end{aligned}$$

Then, we have that $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$. □

Theorem. *If $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \geq 0$, then $\sum a_n$ converges if $L < 1$.*

Remark. This is a sort of generalization of the geometric series.

Proof. Take x such that $L < x < 1$. For $\epsilon = x - L$, $\exists N \mid n \geq N \implies \frac{a_{n+1}}{a_n} < x = L + \epsilon$. Hence $\forall k$, by induction,

$$\frac{a_{N+k}}{a_N} = \frac{a_{N+1}}{a_N} \cdot \frac{a_{N+2}}{a_{N+1}} \cdot \ldots \cdot \frac{a_{N+k}}{a_{N+k-1}} < x^k$$

Thus, $a_{N+k} < x^k a_N$. By comparison, we know that the series $\sum_{k=0}^{\infty} a_{N+k} \leq \sum_{k=0}^{\infty} a_N x^K$, which converges. Then, the partial sums converge and so $\sum_{k=0}^{\infty} a_k$ converges. □

18 Sequences and Series of Functions

Definition. Let $I \subset \mathbb{R}$ be an interval. A sequence of functions is a function $I \times \mathbb{Z}_{\geq 0} \to \mathbb{R}$. We denote $f(x, n)$ by $f_n(x)$. We may start at different indices.

Definition. A sequence of functions $\{f_n\}$ converges pointwise to f if $\forall x \in I, \lim_{n\to\infty} f_n(x) = f(x)$. More pedantically, we have that $\forall x \in I, \forall \epsilon > 0, \exists N \mid n \geq N, |f_n(x) - f(x)| < \epsilon$.

Example. Let $I = [0, 1]$ and $f_n(x) = x^n$. We have that $\{f_n\}$ converges to

$$f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

To check this, if $x = 1, f_n(1) = 1^n = 1$. Otherwise, if $x \neq 1 \implies 0 < x < 1$, and we have shown that $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$, and so x^n must converge to 0.

Definition. A sequence of functions $\{f_n\}$ converges uniformly to f if $\forall \epsilon > 0, \exists N \mid \forall n \geq N, \forall x \in I, |f_n(x) - f(x)| < \epsilon$.

Remark. All sequences which converge uniformly converge pointwise.

Example. It turns out that the previous example is not uniformly convergent (see the following theorem).

Theorem. *If $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ uniformly, then f is also continuous.*

Proof. We need to show that if $y \in I$, then $\forall \epsilon > 0, \exists \delta > 0 \mid |x - y| < \delta$ and $x \in I \implies |f(x) - f(y)| < \epsilon$.

Given $\epsilon > 0$, we know that $\exists N \mid \forall n \geq N, \forall x \in I, |f_n(x) - f(x)| < \frac{\epsilon}{3}$. Further, since f_N is continuous, we know that $\exists \delta > 0 \mid |x - y| < \delta \implies |f_N(x) - f_N(y)| < \frac{\epsilon}{3}$. Therefore, for this particular choice of delta, we know that $|x - y| < \delta \implies |f_n(x) - f(y)| = \epsilon, |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. □

Definition. A series of functions $\{f_n\}$ is the sequence of partial sums $\sum_{i=0}^{n} f_i$.

Corollary. *If $\{f_n\}$ are continuous and $\sum_{n=0}^{\infty} f_n$ uniformly converges to f, then f is continuous.*

Proof. Each of the partial sums, as sums of continuous functions, must be themselves continuous. Thus, f must also be continuous. □

Definition. (Integration of Sequences) If $f_n : [a,b] \to \mathbb{R}$ are continuous, and $f_n \to f$ uniformly, and if $g_n(x) = \int_a^x f_n(t)dt$ and $g(x) = \int_a^x f(t)dt$, then $g_n \to g$ uniformly. (One can exchange $\int$ and uniform lim).

Proof. Given $\forall \epsilon > 0$, $\exists N \mid \forall n \geq N \implies |f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)}$. Then, if $n \geq N$, $|g_n(x) - g(x)| = |\int_a^x f_n - \int_a^x f| = |\int_a^x (f_n - f)| \leq \int_a^x |f_n - f| < \int_a^x \frac{\epsilon}{2(b-a)} < \epsilon$. □

Corollary. *If $\sum_{n=0}^{\infty} f_n \to f$ uniformly, each $f_n : [a,b] \to \mathbb{R}$ is continuous, then*

$$\int_a^x f = \int_a^x \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} \int_a^x f_n$$

Proof. Substitute $h_m = \sum_{n=0}^{m} f_n$ into the above theorem. Since we know that in finite sums, $\int_a^x h_m = \int_a^x \sum_{n=0}^{m} f_n = \sum_{n=0}^{m} \int_a^x f_n$. Thus, we have that $\int_a^x f = \lim_{m\to\infty)} \int_a^x h_m = \lim_{m\to\infty} \sum_{n=0}^{m} \int_a^x f_n = \sum_{n=0}^{\infty} \int_a^x f_n$. □

Theorem. *(Weierstrass M-test) If $f_n : I \to \mathbb{R}$ and $\forall n, \exists M_n \in \mathbb{R}$ such that $\forall x \in I$, $|f_n(x)| \leq M_n$ and $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly (and absolutely) to a limit.*

Proof. Since $\sum_{n=0}^{\infty} M_n = M$ converges, we have that $\forall \epsilon > 0, \exists N \mid n \geq N, |M - \sum_{i=0}^{n} M_i| = |\sum_{i=n+1}^{\infty} M_i| < \epsilon$. Further, $|f(x) - \sum_{i=0}^{n} f_i(x)| = |\sum_{i=n+1}^{\infty} f_n(x)| \leq \sum_{i=n+1}^{\infty} |f_n(x)| \leq |\sum_{i=n+1}^{\infty} M_i| < \epsilon$ by the comparison test for all x, where $f(x) := \sum_{i=0}^{\infty} f_i(x)$, and so $\sum_{i=0}^{n} f_i \to f$ uniformly. □

Example. Take $f_n(x) = \frac{\sin(nx)}{2^n}$. Each of these has $|f_n| \leq \frac{1}{2^n} = M_n$. Then, we have that $\sum_{n=0}^{\infty} \frac{\sin(nx)}{2^n}$ is continuous!

19 Power Series

Definition. If $\{a_n\}$ is a sequence of reals, then the series $\sum_{n=0}^{\infty} a_n x^n$ is called the power series corresponding to that sequence centered at 0. $\sum_{n=0}^{\infty} a_n (x-c)^n$ is the same, but centered instead at c.

Lemma. *Assume a power series centered at c converges at x. Then it converges absolutely for all $y \in \mathbb{R}$ such that $|y - c| < |x - c|$.*

Proof. Assume $x = c$ and $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges. The divergence test yields that $\lim_{n\to\infty} a_n(x-c)^n = 0$. Then, $\exists N \mid \forall n \geq N |a_n(x-c)^n| < 1$. Set $z = \frac{|y-c|}{|x-c|} < 1$.

Then, if $n \geq N$, $|a_n(y-c)^n| = |a_n(x-c)^n| z^n < z^n$. Then, by comparison test, we have that since $0 \leq z < 1$, $\sum_{n=0}^{\infty} z^n$ converges and so $\sum_{n=0}^{\infty} |a_n(y-c)^n|$ converges. □

Theorem. *For any power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$, *exactly one of the following occurs:*

1. *It converges absolutely everywhere*
2. *It converges absolutely only at* $x = c$
3. $\exists R > 0$ *such that the series converges absolutely on* $(c-R, c+R)$ *and diverges elsewhere*

Proof. Clearly 1), 2), 3) are mutually exclusive.

Now, assume that 1), 2) do not happen. Let $S = \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x-c)^n$ absolutely converges $\}$. Note that $c \in S$. If $x \in S$ and $r = |x-c|$, then the lemma implies that $(c-r, c+r) \subseteq S$. Further, if $x \notin S$, then $S \subseteq [c-r, c+r]$. Since 1) is false, there is a point where it does not converge, and so S is bounded, and so $\sup(S)$ exists. Let $R := \sup(S) - c$. Then, since 2) is false, $R > 0$.

If $|x-c| > R$, then $\exists y \mid y - c \in (R, |x-c|)$, and so $y > c + R = \sup(S) \implies y \notin S$. Further, $x \notin S$. Hence, $S \subseteq [c-R, c+R]$.

On the other hand, if $|x-c| < R, \exists y \mid (c + |x-c|, \sup(S))$ by the approximation property, and so $|x-c| < y - c$, and since $x \in S$, $(c-R, c+R) \subseteq S$. □

Remark. The case of 3), we say nothing about $x = c - R$ or $x = c + R$. This R is also called the radius of convergence. In 1), we say $R = \infty$, and in 2), $R = 0$.

Definition. Let $0! = 1$. If $n \in \mathbb{Z}_{>0}, n! = (n-1)! \cdot n$.

Example. The power series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converge everywhere.

Definition.

$$\sin(x) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

Definition.

$$\cos(x) := \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Remark. We have that $\sin(0) = 0, \cos(0) = 1$, as well as $\sin(-x) = \sin(x), \cos(-x) = \cos(x)$.

Prop. *Let* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *be a power series with* $R > 0$ *and let* $[a, b] \subset (c-R, c+R)$. *Then the partial sums converge uniformly on* $[a, b]$.

Proof. $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges if $x \in [a,b]$. Then, we have that $\lim_{n\to\infty} a_n(x-c)^n = 0$. Further, $\exists y$ with $y \in (c-R, c+R), |y-c| > D = \max(|a-c|, |b-c|)$. Let $z = \frac{D}{|y-c|} < 1$. Then, $|a_n(x-c)^n| \leq |a_n D^n| \leq |a_n(y-c)^n| z^n \leq z^n$, for $n \geq N$ for some N. Let $M_n = z^n$, and the M test yields uniform convergence. □

Corollary. *$\sum_{n=0}^{\infty} a_n(x-c)^m$ is continuous in the interval of convergence.*

Proof. This follows from the continuity of polynomials. □

Remark. It turns out that power series are always really really nice, even nicer than random infinitely differentiable functions!

Theorem. *Assume that $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ (f is given by a power series on $(c-R, c+R)$. Then,*

1. $\int_c^x f = \sum_{n=0}^{\infty} \int_c^x a_n(t-c)^n dt = \sum_{n=0}^{\infty} \frac{a_n(t-c)^{n+1}}{n+1}$
2. *f is differentiable in $(c-R, c+R)$ and* $f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n(x-c)^n) = \sum_{n=1}^{\infty} a_n n(x-c)^{n-1}$

Proof. 1) follows from an earlier general result.

For 2), assume that $c = 0$ (in the nonzero case, you just tack on $-c$ everywhere). Consider $\sum_{n=0}^{\infty} a_n n x^{n-1}$. We want to show that this sum converges in $(-R, R)$.

First assume that x is positive. Pick an h such that $x < x+h < R$. Then, $\frac{f(x+h)-f(x)}{h} = \sum a_n \frac{(x+h)^n - x^n}{h}$. Then Mean Value Theorem applied to x^n in $(x, x+h)$ yields that $\exists c_n \mid x < c_n < x+h$ and $\sum_{n=1}^{\infty} a_n n c_n^{n-1} = \frac{f(x+h)-f(x)}{h}$. Further, we have that a termwise comparison test has that the original series converges. The other side is the same, more or less. Let $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. The theorem about term-by-term integration gives that $\int_0^x g(t)dt = \sum_{n=1}^{\infty} a_n x^n + C$. Since we have that g is continuous, we have by the Fundamental Theorem of Calculus that $f'(x)$ exists and equals $g(x)$. □

Definition. A function is smooth when it is infinitely differentiable.

Remark. All power series that converge are smooth.

Example.

$$(\cos(x))^2 + (\sin(x))^2 = 1$$

Proof. Let $h(x) = (\cos(x))^2 + (\sin(x))^2$. Consider that $h'(x) = -2\sin(x)\cos(x) + 2\sin(x)\cos(x) = 0$, and that $h(0) = 1$, so we have that $h(x) = 1$. □

Example.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Proof. Since we have that power series are smooth,

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{1}{x}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}=\lim_{x\to 0}\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n}}{(2n+1)!}=1$$

□

Remark. In general, one can compute limits inside power series at the center of the power series with this method.

20 Taylor Series

Remark. Say that we have a power series $f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n \implies f(c)=a_0$. Further, $f'(x)=\sum_{n=1}^{\infty}a_n n(x-c)^{n-1} \implies f'(c)=a_1$ and $f''(x)=\sum_{n=1}^{\infty}a_n n(n-1)(x-c)^{n-2} \implies 2a_2$. Similarly, we have that $f^k(c)=k!a_k$. This is important:

$$a_k=\frac{f^k(c)}{k!} \implies f(x)=\sum_{n=0}^{\infty}\frac{f^k(c)}{k!}(x-c)^k$$

Corollary. *Power series are unique.*

Example. What is the Taylor Series of e^x at $c=0$? We have that $a_n=\frac{\exp^n(0)}{n!}=\frac{\exp(0)}{n!}=\frac{1}{n!}$. Then, the Taylor Series is $\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Proof. A function on some interval I is analytic if $\forall c\in I,\exists R>0 \mid f$ is represented by a power series (the Taylor Series, specifically, since we know that it has to be unique) in this interval $(c-R,c+R)$. □

Prop. *Sums, differences, products, quotients (if the denominator is nonzero) compositions of smooth / analytic functions are smooth / analytically respectively.*

Proof. The only difficult ones are analytic; these are omitted. □

Definition. If $f:I\to R$ is smooth, its Taylor Series at $c\in I$ is

$$\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n$$

Example. The following is not analytic:

$$f(x)=\begin{cases} e^{\frac{-1}{x^2}} & x\neq 0\\ 0 & x=0\end{cases}$$

Theorem. *(Taylor's Theorem) Let f be smooth on some interval $(c-R, c+R)$. Then, $\forall N \geq 0$, we have that $f(x) = \sum_{n=0}^{N} \frac{f^n(c)}{n!}(x-c)^n + E_N(x)$, where $E_N(x) = \frac{1}{N!}\int_c^x (x-t)^N f^{N+1}(t)dt$. Further, the Taylor Series of f converges at $x \iff E_N(x) \to 0$ as $N \to \infty$.*

Proof. Induct on N. $N = 0$ is the statement $f(x) = f(c) + \int_c^x f'(t)dt$. This follows from the Fundamental Theorem of Calculus.

Assume that this is known for N. Then, with integration by parts,

$$\begin{aligned}
E_{N+1}(x) &= E_N(x) - \frac{f^{N+1}(c)}{(N+1)!}(x-c)^{N+1} \\
&= \frac{1}{N!}\int_c^x (x-t)^N f^{N+1}(t)dt - \frac{f^{N+!}(c)}{N!}\int_c^x (x-t)^N dt \\
&= \frac{1}{N!}\int_c^x (x-t)^N (f^{N+1}(t) - f^{N+1}(c))dt \\
&\quad v = \frac{-(x-t)^{N+1}}{N+1} \\
&\quad du = f^{N+2}(t) - f^{N+2}(c) \\
&= uv|_{t=c}^{t=x} - \frac{1}{N!}\int_c^x vdu \\
&= \frac{1}{(N+1)!}\int_c^x (x-t)^{N+1} f^{N+2}(t)dt
\end{aligned}$$

$\square$

Prop. *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ have radius of convergence R. If $y \in (c-R, c+R)$, then the Taylor Series of f with center y has radius of convergence at least $\min(|y-c+R|, |y-c-R|)$.*

Proof.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n \\
&= \sum_{n=0}^{\infty} a_n (x-y+y-c)^n \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (x-y)^{n-k}(y-c)^k \\
&\leq \sum_{n=0}^{\infty} \left| a_n \sum_{k=0}^{n} \binom{n}{k} (x-y)^{n-k}(y-c)^k \right| \\
&\leq \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |x-y|^{n-k}|y-c|^k \\
&= \sum_{n=0}^{\infty} |a_n||z-c|^n, \text{ where } z = |x-y| + |y-c| + c
\end{aligned}$$

This converges when $c - R < z < c + R$. By comparison, we have that $\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (x-y)^{n-k}(y-c)^k$ is also convergent, allowing us to rearrange and giving

$$\sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (x-y)^{n-k} \right) (y-c)^k$$

The condition that $c - R < z < c + R$ holds exactly when we want it to, i.e. when x is within $\min(|y-c+R|, |y-c-R|)$ of y. □

Corollary. *f is analytic on $(c-R, c+R)$.*

Example. $\exp(x)$ is analytic. The Taylor Series at 0 is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. By Taylor's Theorem, we have that we need $E_N(x) \to 0$.

$$\begin{aligned}
E_N(x) &= \frac{1}{N!} \int_0^x (x-t)^N e^t dt \\
&\leq \frac{1}{N!} x^N e^x x
\end{aligned}$$

which goes to zero as $N \to \infty$. This, since TS at 0 is $\exp(x) \forall x \in R$, the proposition implies that exp is analytic.

Example. Since when $|x| < 1$, we have that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, plugging in x^2 for x gives that $\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$ when $|x| < 1$.

Similarly, $\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}$ when $|x| < 1$. Integrating, we see that $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ when $|x| < 1$. The logarithm is analytic everywhere, but this in particular shows that it is analytic on $(1,2)$.

Example.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

when $|x| < 1$.

Theorem.

$$e \notin \mathbb{Q}$$

Proof. We know that $\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$. It suffices to show that $\frac{1}{e} \notin \mathbb{Q}$. Let s_n be the n^{th} partial sum of the above.

We know that the error at the n^{th} step is bounded by the absolute value of the $n+1^{th}$ term itself. Then, we have that $0 < \frac{1}{e} - s_{2n-1} < \frac{1}{(2n)!}$. Then,

$$0 < (2n-1)!(\frac{1}{e} - 2_{2n-1}) < \frac{1}{2n} \leq \frac{1}{2}$$

Now let $\frac{1}{e} = \frac{p}{q}, p, q \in \mathbb{Z}$. Now, pick $n \geq \frac{q+1}{2}$. Then, we have that $(2n-1)!\frac{1}{e}$ is an integer. Similarly, we have that $(2n-1)!s_{2n-1}$ is also an integer, yielding an integer between 0 and $\frac{1}{2}$. $\square$